Chapter 3
Determinant and Inverse of Matrix

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2006-03-21

Transpose of a matrix

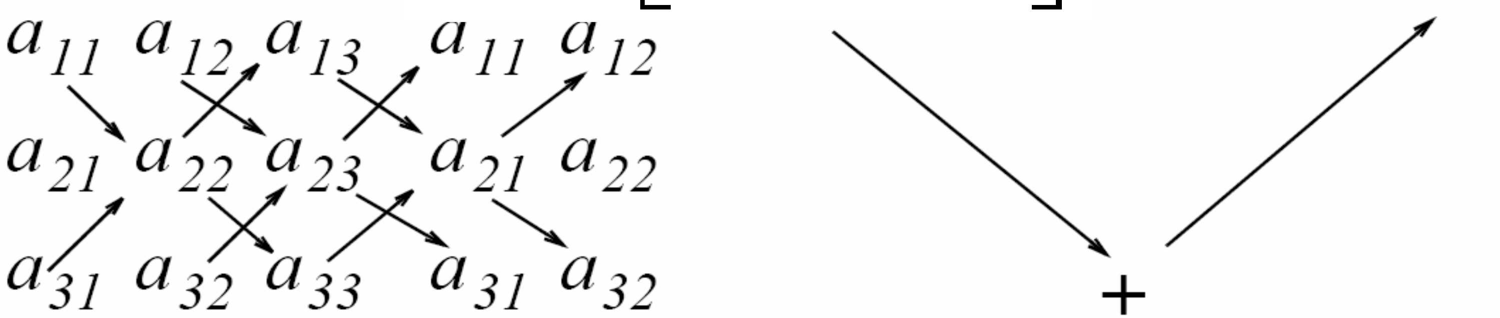
- The transpose of a matrix A that is obtained when the rows and columns of matrix are interchanged :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Algorithm of determinant computation

- Let matrix A be defined as the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



Ex.1

- Compute *det A* and *det B* given that

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -3 & -4 \\ 1 & 0 & -2 \\ 0 & -5 & -6 \end{bmatrix}$$

*Ans: det A =*9, *det B =-18*

Minors and Cofactors

- Remove the elements of its ith row, and jth column, the determinant of the remaining determinant of the remaining n-square matrix is called the *minor of determinant A,* denoted as $[M_{ij}]$
- The signed minor α_{ij} is called the *cofactor* of a_{ij} *:*

$$(-1)^{i+j}[M_{ij}]$$

Determinant of a matrix of order *4* or higher

- A fourth-order determinant can first be expressed as the sum of the products of the elements of its first row by its cofactor

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{11}\begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} - a_{21}\begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$+ a_{31}\begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} - a_{41}\begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Ex.2 Compute the value of the determinant

$$A = \begin{bmatrix} 2 & -1 & 0 & -3 \\ -1 & 1 & 0 & -1 \\ 4 & 0 & 3 & -2 \\ -3 & 0 & 0 & 1 \end{bmatrix}$$

Download the source code:

Determinant.cpp

Ans: det A = -33

Cramer's Rule

$$a_{11}x + a_{12}y + a_{13}z = A$$
$$a_{21}x + a_{22}y + a_{23}z = B$$
$$a_{31}x + a_{32}y + a_{33}z = C$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad D_1 = \begin{vmatrix} A & a_{11} & a_{13} \\ B & a_{21} & a_{23} \\ C & a_{31} & a_{33} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{11} & A & a_{13} \\ a_{21} & B & a_{23} \\ a_{31} & C & a_{33} \end{vmatrix} \qquad D_3 = \begin{vmatrix} a_{11} & a_{12} & A \\ a_{21} & a_{22} & B \\ a_{31} & a_{32} & C \end{vmatrix}$$

- *Cramer's rule* states that the unknowns *x, y,* and *z* can be found from the relations

$$x = \frac{D_1}{\Delta} \qquad y = \frac{D_2}{\Delta} \qquad z = \frac{D_3}{\Delta}$$

Ex.3

Use Cramer's rule to find *v1*, *v2* and *v3*, if

$$2v_1 - v_2 + 3v_3 = 5$$
$$-4v_1 - 3v_2 - 2v_3 = 8$$
$$3v_1 + v_2 - v_3 = 4$$

Ans:

$$x_1 = \frac{D_1}{\Delta} = \frac{85}{35} = \frac{17}{7}$$

$$x_2 = \frac{D_2}{\Delta} = -\frac{170}{35} = -\frac{34}{7}$$

$$x_3 = \frac{D_3}{\Delta} = -\frac{55}{35} = -\frac{11}{7}$$

Adjoint of a Matrix

- *the adjoint of* square matrix A, and α *ij* is is the cofactor of *aij, is defined*

$$adj A = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} & \dots & \alpha_{n2} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \dots & \alpha_{n3} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{1n} & \alpha_{2n} & \alpha_{3n} & \dots & \alpha_{nn} \end{bmatrix}$$

Example of Adjoint of a Matrix

- Compute *adjA given that* $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$

$$adjA = \begin{bmatrix} \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

Inverse of a Matrix

If and *B* are square matrices such that $AB=BA=I$, where I is the identity matrix, *B* is called the *inverse* of *A*. denoted as $B=A^{-1}$, and likewise, $A=B^{-1}$.

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

Ex.4

Find A^{-1}, given that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$$